the other hand, by reversing the direction of the vector d previously given, one has the completely symmetrical configuration that the line-of-sight vectors are all separated by the same angle $\cos^{-1}(-1/3)$. For this configuration, one may compute the eigenvalues of HH^T as $\lambda_1=\lambda_2=\lambda_3=4/3$ and $\lambda_4=4$, giving rise to a navigation performance index $\mathrm{tr}(HH^T)^{-1}=2.5$ Notice that $\lambda_1=4/3$ achieves the upper bound $1-\cos\beta$ discussed earlier. Since any perturbation of the configuration will result in a decrease in the minimum angle between two line-of-sight vectors, and therefore a decrease in λ_1, this configuration maximizes the smallest eigenvalue of HH^T. Whether this also happens to be the best configuration remains to be investigated.

Acknowledgments

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References

[1] Pearson, C.E., (ed.), *Handbook of Applied Mathematics*, 1st Ed., Van Nostrand, New York, 1974, p. 926.

[2] Householder, A.S., *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953, pp. 78-79.

Technical Comments

AIAA 81-4006

Comment on "Formulation of Equations of Motion for Complex Spacecraft"

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KANE and Levinson[1] have written a very comprehensive paper which compares various techniques for formulating the equations of motion for complex systems of moving masses. In general their analysis is quite commendable, however, there are two points worthy of comment: 1) In formulating the angular momentum equations it is not necessary to locate explicitly the combined mass center (CMC) for the entire system. Thus by doing a few preliminary vector calculations, Eq. (10KL)† can be derived directly rather than arriving at Eq. (9KL) and then algebraically simplifying this to Eq. (10KL). 2) The origin for the angular momentum equation is not restricted to the CMC; any origin can be used provided an additional term is added. This "generalized angular momentum equation" effectively constitutes the proof of D'Alembert's principle. These points will now be elaborated upon.

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†Equation or Fig. numbers followed by KL refer to those in Ref. 1.

‡Since there is only a single mass m moving relative to the main body, the "1" subscript can be dropped.

1) The angular momentum for a system of bodies with respect to their CMC can be written as the sum of the angular momenta for each body with respect to its own mass center (MC) plus transfer terms of the form

$$m_r\bar{\rho}_r\times\dot{\bar{\rho}}_r \qquad (r=0,1,2,\ldots,n) \tag{1}$$

where m_r is the mass of body r (there are $n+1$ bodies), $\bar{\rho}_r$ is the position vector of the MC of body r relative to the CMC, and $\dot{\bar{\rho}}_r=\mathrm{d}\bar{\rho}_r/\mathrm{d}t$, where $\mathrm{d}/\mathrm{d}t$ is the time derivative taken with respect to a nonrotating coordinate system centered at the CMC. In these "moving parts" type problems $\bar{\rho}_r$ is usually specified with respect to the MC of the main body B_0 by a relative position vector $\bar{q}_r$. Thus if $\bar{\rho}_0$ is the position vector from the CMC to the MC of B_0

$$\bar{\rho}_r=\bar{\rho}_0+\bar{q}_r \tag{2}$$

By definition of the CMC

$$\sum_{r=0}^{n} m_r\bar{\rho}_r=0 \tag{3}$$

Eliminating $\bar{\rho}_0$ between Eqs. (2) and (3) determines $\bar{\rho}_r$ (and $\dot{\bar{\rho}}_r$) as

$$\bar{\rho}_r=\bar{q}_r-\frac{1}{\Sigma m}\sum_{r=1}^{n} m_r\bar{q}_r \qquad (r=0,1,2,\ldots,n;\ \bar{q}_0\equiv 0) \tag{4}$$

where

$$\Sigma m=\sum_{r=0}^{n} m_r$$

is the system total mass. For the KL example the transfer terms are, from Eq. (1),

$$M\bar{\rho}_0 \times \dot{\bar{\rho}}_0 + m\bar{\rho}_1 \times \dot{\bar{\rho}}_1 \tag{5}$$

Using the KL notation, let the motion of mass m (point P) relative to the MC of main body A (point A^*) be defined by $\bar{q}$‡ where

$$\bar{q} = \bar{a}_1 q_1 + \bar{a}_2 q_2 \tag{6}$$

and where, referring to Fig. 1KL,

$$q_1 = b\sin\theta \qquad q_2 = c - r - b\cos\theta \tag{7}$$

From Eqs. (4)

$$\bar{\rho}_0 = -(\mu/M)\bar{q} \qquad \bar{\rho}_1 = (\mu/m)\bar{q} \tag{8}$$

where $\mu = mM/(M+m)$. Combining Eqs. (5) and (8) the total transfer term is $\mu\bar{q} \times \dot{\bar{q}}$ so that the total angular momentum with respect to the CMC is

$$\bar{H}_{\mathrm{CMC}} = \bar{a}_1 I_1 \omega_1 + \bar{a}_2 I_2 \omega_2 + \bar{a}_3 I_3 \omega_3 + \mu\bar{q} \times \dot{\bar{q}} \tag{9}$$

Forming the inertial time derivative of $\bar{H}_{\mathrm{CMC}}$ and equating this to the vector torque around the CMC produces Eqs. (10-12KL) directly. Equation (9KL) and its counterparts never appear.

2) A generalized angular momentum equation was derived by the author in a 1960 article[2] and subsequently applied in a 1962 paper.[3] An alternate derivation starting from first principles was later given in an internal set of notes.[4] Let point 0 be some reference point, moving or fixed, of any system of moving masses. Then with respect to this point the generalized equation is

$$\bar{L}_0 = \dot{\bar{H}}_0 + \bar{S}_0 \times \bar{a}_0 \tag{10}$$

In Eq. (10) $\bar{L}_0$ is the torque resulting from external forces, $\bar{H}_0$ is the system angular momentum, $\bar{S}_0$ is the system static moment (all evaluated with respect to point 0), and $\bar{a}_0$ is the inertial acceleration of point 0. For the KL example, take the reference point as the MC of the main body, point A^*. Evaluating all terms with respect to A^* and using the KL notation, actual or implied,

$$\bar{L}_{A^*} = \bar{T} + \bar{q} \times \bar{R} \qquad \bar{S}_{A^*} = m\bar{q}$$

$$\bar{H}_{A^*} = \bar{a}_1 I_1 \omega_1 + \bar{a}_2 I_2 \omega_2 + \bar{a}_3 I_3 \omega_3 + m\bar{q} \times \dot{\bar{q}} \tag{11}$$

The inertial acceleration of point A^* can be expressed in two ways: 1) kinematically, by introducing the velocity $\bar{v}^{A^*}$, and 2) dynamically, by introducing the external forces.

1) Since the inertial velocity is expressed in components along rotating axes, Eq. (3KL), then

$$\bar{a}^{A^*} = \bar{a}_1 \dot{v}_1 + \bar{a}_2 \dot{v}_2 + \bar{a}_3 \dot{v}_3 + \bar{\omega} \times \bar{v}^{A^*} \tag{12}$$

Substituting Eq. (12) for $\bar{a}^{A^*}$, combining with Eq. (11) for $\bar{S}_{A^*}$, and combining the result with $\bar{H}_{A^*}$ and $\bar{L}_{A^*}$ produces Eqs. (23-25KL).

2) Dynamically

$$\frac{\mathrm{d}}{\mathrm{d}t}(M\bar{v}^{A^*} + m\bar{v}^P) = M\bar{a}^{A^*} + m\bar{a}^P = \bar{R} + \bar{S} \tag{13}$$

where $\bar{a}^P = \bar{a}^{A^*} + \ddot{\bar{q}}$, and where $\ddot{\bar{q}}$ is the inertial second derivative of $\bar{q}$ defined by Eqs. (6) and (7). Solving Eq. (13) for $\bar{a}^{A^*}$ and then combining with $\bar{S}_{A^*}$ and the result with $\bar{H}_{A^*}$ and $\bar{L}_{A^*}$ produces again Eqs. (10-12KL).

References

[1] Kane, T.R. and Levinson, D.A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, March-April 1980, pp. 99-112.

[2] Grubin, C., "On Generalization of the Angular Momentum Equation," *Journal of Engineering Education*, Vol. 51, 1960, p. 237.

[3] Grubin, C., "Dynamics of a Vehicle Containing Moving Parts," *Journal of Applied Mechanics*, Sept. 1962, pp. 486-488.

[4] Grubin, C., *Mechanics*, West Coast University, Los Angeles, Calif., 1967, Chap. 6.

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Reply by Authors to C. Grubin

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WE believe that the first of Dr. Grubin's points is factually incorrect and that the second is misleading. Accordingly, we shall attempt to set matters straight by discussing each point in turn.

1) Dr. Grubin's assertion that "by doing a few preliminary vector calculations, Eq. (10KL) can be derived directly rather than arriving at Eq. (9KL) and then ..." is invalid. This may be seen as follows.

The two equations at issue are

$$\begin{aligned} I_1\dot{\omega}_1 &- (I_2 - I_3)\omega_2\omega_3 + \mu\langle (b\dot{\theta}\sin\theta - \dot{r})[\omega_1(c - r - b\cos\theta) \\ &- \omega_2 b\sin\theta] + (c - r - b\cos\theta)[\dot{\omega}_1(c - r - b\cos\theta) \\ &+ \omega_1(b\dot{\theta}\sin\theta - \dot{r}) - \dot{\omega}_2 b\sin\theta - \omega_2\dot{\theta}b\cos\theta] \\ &+ \omega_2[b^2\dot{\theta} + b\sin\theta(\omega_3 b\sin\theta - \dot{r}) - (c - r)b\dot{\theta}\cos\theta \\ &+ (c - r - b\cos\theta)^2\omega_3] + \omega_3 b\sin\theta[\omega_1(c - r - b\cos\theta) \\ &- \omega_2 b\sin\theta]\rangle = T_1 + \mu(c - r - b\cos\theta)(R_3/m - S_3/M) \end{aligned} \tag{9KL}$$

and

$$\begin{aligned} I_1\dot{\omega}_1 &- (I_2 - I_3)\omega_2\omega_3 + \mu(c - r - b\cos\theta)[(\dot{\omega}_1 \\ &+ \omega_2\omega_3)(c - r - b\cos\theta) - (\dot{\omega}_2 - \omega_3\omega_1)b\sin\theta + 2b\dot{\theta}(\omega_1\sin\theta \\ &- \omega_2\cos\theta) - 2\dot{r}\omega_1] = T_1 + \mu(c - r - b\cos\theta)(R_3/m - S_3/M) \end{aligned} \tag{10KL}$$

In our paper, it is shown that use of the Angular Momentum Principle leads directly to the first of these, and it is asserted that a considerable amount of labor is required to deduce the second from the first. Let us carry out the procedure which,

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